Microeconomics

Ph.D. Exam 15/03/2017
Solutions for questions in Section B

1 Question 1

Consider the following game in strategic form:

		P2			
		W	X	Y	Z
P1	A	$1,2$	$2,1$	$3,5$	$4,2$
	B	$5,3$	$0,4$	$0,0$	$5,7$
	C	$3,1$	$1,2$	$3,1$	$3,0$
	D	$2,3$	$4,3$	$2,0$	$2,0$

- Find the equilibrium of the game using the iterated deletion of dominated strategies.

  Solution. We start from the definition of a dominated strategy. Given two strategies s_i and $s_i' \in S_i$ we say that s_i' is strictly dominated if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \,\forall\; s_{-i} \in S_{-1}$. In words, a strategy is strictly dominated if it generates a payoff which is smaller than the one that would be obtained if another strategy is chosen, given any possible strategy selected by the opponent. A strategy can also be weakly dominated if the associated payoff is not smaller and in some cases stricly smaller than the one obtained using another strategy.

  In the game above none of the two players has a dominanted strategy. Let us consider for example Player 1. Strategy A is not dominated by any other one. As an example, let us compare the payoffs that Player 1 obtains if he/she chooses B instead of A. Player 1 obtains $1 < 5$ if Player 2 chooses W; $4 < 5$ if Player 2 chooses Z. However if the latter chooses one of the other two strategies (X or y), choosing A makes Player 1 better off, since he/she obtains $2 > 0$ and $3 > 0$, respectively. Making similar comparisons for all the strategies available to Player 1 and 2, we cannot find any strategy, which makes one of them better off, no matter opponent's choice. Hence, we cannot define any equilibrium using iterated deletion of dominated strategies.

- Explain the Nash equilibrium solution concept and check whether there exists a Nash equilibrium in pure strategy. Explain carefully the steps you take. (**Hint**: Do not write down simply your solution, but explain how you did reach your answer). Discuss the welfare implication of the equilibrium/equlibria you found, if any.

Solution. Fix the strategy of one player and check which is the best response of the opponent. For instance, let us assume that Player 1 plays A. In this case, player 2 chooses to play Y, since it gives the higher payoff. Suppose, instead, that Player 1 plays B. In this case the best response for Player 2 is to play Z. Running the same check on the remaining strategies for Player 1 and developing the same arguments for Player 2, it is easy to notice that the game has three possible equilibria in pure strategy, i.e. $[\{D, X\}; \{A, Y\}; \{B, Z\}]$.

- Find at least one Nash equilibrium in mixed strategies (**HINT**: Besides a pure strategy equilibrium, which is played with probability 1, the game contains 6 mixed strategy equilibria. Some of them assume that a player chooses not to play one or more strategies. The easiest Nash equilibrium in mixed strategy that you can find is the one in which both players randomize over **all** available strategies).

Solution. Finding an equilibrium in mixed stategies implies for each player to find a probability distribution over the possible strategies of his/her opponent such that the former is indifferent among his/her available strategies. A game can have more than one equilibrium in mixed strategy. Also, notice that any equilibrium in pure strategy is equivalent to an equilibrium in mixed stragies where every player plays a given strategy with probability 1.

Let us assume that Player 1 randomizes over his strategy with probability p_i $\forall i = A, B, C, D$, while Player 2 assigns a probability q_i $\forall i = W, X, Y, Z$ to each of her strategies, i.e.:

		P2			
		(q_W) W	(q_X) X	(q_Y) Y	(q_Z) Z
P1	(p_A) A	$1, 2$	$2, 1$	$3, 5$	$4, 2$
	(p_B) B	$5, 3$	$0, 4$	$0, 0$	$5, 7$
	(p_C) C	$3, 1$	$1, 2$	$3, 1$	$3, 0$
	(p_D) D	$2, 3$	$4, 3$	$2, 0$	$2, 0$

To find the appropriate values of the above probabilities, we need to assume that a player is indifferent between choosing one of his own strategies, conditional on the choice made by its opponent. For instance, let us calculate Player 1's expected payoff from playing his avaliable strategies:

$$
\begin{aligned}
E_1(A) &= 1q_W + 2q_X + 3q_Y + 4q_Z \\
E_1(B) &= 5q_W + 0q_X + 0q_Y + 5q_Z \\
E_1(C) &= 3q_W + 1q_X + 3q_Y + 3q_Z \\
E_1(D) &= 2q_W + 4q_X + 2q_Y + 2q_Z
\end{aligned}
$$

Notice that in equilibrium Player 1 should be indifferent between playing one of the available strategies. Therefore in order to find the probabilities $q_i \ \forall i = W, X, Y, Z$, which make him indifferent among his own strategies, we should satisfy the following equalities:

$$\begin{aligned} E_1(A) = E_1(B) = E_1(C) = E_1(D) \\ q_W + q_X + q_Y + q_Z = 1 \end{aligned}$$

Let us start by equalizing $E_1(A)$ and $E_1(C)$:

$$\begin{aligned} E_1(A) &= E_1(C) \\ 3q_W + q_X + 3q_Y + 3q_Z &= q_W + 2q_X + 3q_Y + 4q_Z \\ 3q_W + q_X + 3q_Z &= q_W + 2q_X + 4q_Z \\ q_X &= 2q_W - q_Z \qquad (1) \end{aligned}$$

We now equalize $E_1(B)$ and $E_1(C)$, making use of the previous result as well:

$$\begin{aligned} E_1(B) &= E_1(C) \\ 3q_W + q_X + 3q_Y + 3q_Z &= 5q_W + 0q_X + 0q_Y + 5q_Z \\ 3q_W + 2q_W - q_Z + 3q_Y + 3q_Z &= 5q_W + 5q_Z \\ 3q_Y + 2q_Z &= 5q_Z \\ 3q_Z &= 3q_Y \\ q_Z &= q_Y \qquad (2) \end{aligned}$$

Finally, we equalize $E_1(C)$ and $E_1(D)$ and substitute Eq. (1) for q_X:

$$\begin{aligned} E_1(C) &= E_1(D) \\ 3q_W + q_X + 3q_Y + 3q_Z &= 2q_W + 4q_X + 2q_Y + 2q_Z \\ 3q_W + 2q_W - q_Y + 3q_Y + 3q_Y &= 2q_W + 4(2q_W - q_Y) + 2q_Y + 2q_Y \\ 3q_W + 2q_1 - q_Y + 3q_Y + 3q_Y &= 10q_W \\ 5q_W + 5q_Y &= 10q_W \\ q_Y &= q_W \qquad (3) \end{aligned}$$

Making use of Eqs. (1) to (3), it is easy to show that:

$$q_W = q_X = q_Y = q_Z$$

Hence, it immediately follows that:

$$q_W = q_X = q_Y = q_Z = \frac{1}{4}$$

Now we consider which are the probabilities over Players 1's strategies, which make Player 2 indifferent on the strategy to play. First, we consider the expected payoff for each strategy:

$$\begin{aligned}
E_2(W) &= 2p_A + 3p_B + 1p_C + 3p_D \\
E_2(X) &= 1p_A + 4p_B + 2p_C + 3p_D \\
E_2(Y) &= 5p_A + 0p_B + 1p_C + 0p_D \\
E_2(Z) &= 2p_A + 7p_B + 0p_C + 0p_D
\end{aligned}$$

First we equalize the expected payoff from playing Y and Z:

$$\begin{aligned}
E_2(Y) &= E_2(Z) \\
5p_A + 0p_B + p_C + 0p_D &= 2p_A + 7p_B + 0p_C + 0p_D \\
5p_A + p_C &= 2p_A + 7p_B \\
3p_A &= 7p_B - p_C \\
p_A &= \frac{7}{3}p_B - \frac{1}{3}p_C \qquad (4)
\end{aligned}$$

We now equalize the expected payoffs from playing W and X:

$$\begin{aligned}
E_2(W) &= E_2(X) \\
2p_A + 3p_B + p_C + 3p_D &= p_A + 4p_B + 2p_C + 3p_D \\
p_A &= p_B + p_C \qquad (5)
\end{aligned}$$

Equalizing Eqs. (4) and (5) yeilds:

$$\begin{aligned}
\frac{7}{3}p_B - \frac{1}{3}p_C &= p_B + p_C \\
\frac{7}{3}p_B - p_B &= \frac{4}{3}p_C \\
p_B &= p_C \qquad (6)
\end{aligned}$$

Eqs. (5) and (6) clearly imply that:

$$p_A = 2p_B = 2p_C \tag{7}$$

We now equalize $E_2(W) = E_2(Z)$. This yields:

$$\begin{aligned}
2p_A + 3p_B + p_C + 3p_D &= 2p_A + 7p_B + 0p_C + 0p_D \\
7p_B + p_C + 3p_D &= 2p_A + 7p_B \\
p_C + 3p_D &= 2p_A \\
p_B + 3p_D &= 4p_B \\
3p_D &= 3p_B \\
p_B &= p_D \qquad (8)
\end{aligned}$$

Using Eqs. (5) to (8) and the trivial fact that the probabiilities must sum up, we have:

$$p_A + \frac{1}{2}p_B + \frac{1}{2}p_C + \frac{1}{2}p_D = 1$$

Hence,

$$p_A + \frac{1}{2}p_A + \frac{1}{2}p_A + \frac{1}{2}p_A = 1 \tag{9}$$

Simplyfing the above equation yields: $p_A = \frac{2}{5}$ and $p_B = p_C = p_D = \frac{1}{5}$.
Hence a strategy profile which can be supported as Nash equlibrium is $\left[\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right); \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right]$.